Statistics C206B Lecture 1 Notes

Daniel Raban

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1 Examples of Random Curves

1.1 Random interfaces

We will consider models of random growth, crystals, existence of multiple phases in various settings of statistical mechanics. In many such examples, the model will be encoded by a random interface (curve in $d = 1$, surface in $d \geq 2$). Usually, such surfaces will be given by a Gibbs measure. The static aspect usually involves understanding the Gibbs measure. A Gibbs measures is naturally associated with a Markov chain (Glauber dynamics or heat bath dynamics). The dynamical aspect involves understanding the Markov chain.

We will encounter:

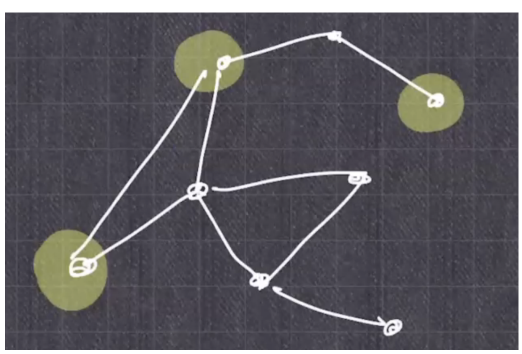
- Hydrodynamic limits under suitable scaling of space and time
- Relaxation to equilibrium

We will begin by looking at some natural examples where we see interfaces appearing.

1.2 Exclusion processes

One example from interacting particle systems is exclusion processes.

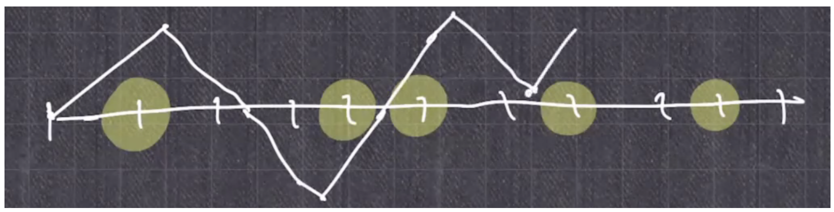
Example 1.1. Take any graph, and suppose we have particles at some of the vertices of the graph:



Particles independently perform a simple random walk on the graph (in continuous or discrete time, but let us focus on continuous time), respecting the exclusion constraint that at every site, there can be at most 1 particle.

We will consider exclusion on $\mathbb{Z}$:

This is known as the **simple symmetric exclusion process (SSEP)**. Given a particle configuration, one can construct a height function in the following way: Whether the particle is at a site or not determines whether the increment of the function is $+1$ or -1.



We should think of the particles as living on the midpoints of each edge. How does the height function evolve according to the dynamics? A particle moving to the right turns a peak into a valley, and a particle moving to the left turns a valley into a peak.

In this example on a segment of $\mathbb{Z}$ with 8 sites, nearest neighbor paths of length 8 which start and end at 0 are in bijection with particle configurations on 8 sites with 4 particles. The stationary measure is uniform over all such paths. As the number of sites goes to ∞, a uniformly chosen scaled random walk bridge looks like a Brownian bridge. The typical maximum height of such a random path is $\sqrt{n}$, and we understand these paths well.

What do the dynamics look like? Look at the expected height $\widetilde{h}_t = \mathbb{E}[h_t]$. How does $\widetilde{h}$ change with time?

$$\widetilde{h}_{t+1}(x) - \widetilde{h}_t(x) = \frac{\widetilde{h}_t(x+1) + \widetilde{h}_t(x-1)}{2} - \widetilde{h}_t(x).$$

If instead of moving time forward by 1, we move it by some small amount, this tells us that $\widetilde{h}_t$ follows a partial differential equation:

$$\frac{d}{dt}\widetilde{h}_t = \frac{\partial^2}{\partial x^2}\widetilde{h}_t(x).$$

So the expected height follows the heat equation.

If we look at the evolution of the actual height function, we get

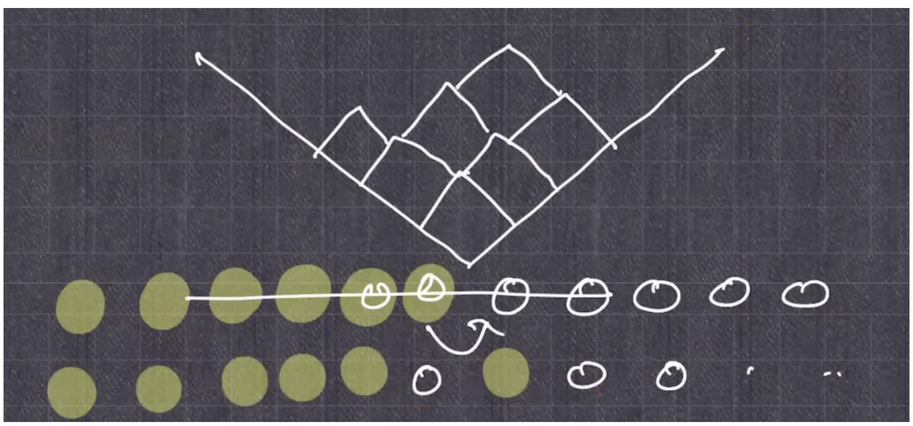
$$\frac{d}{dt}h_t = \frac{\partial^2}{\partial x^2}h_t + \eta(x,t),$$

where η is space time white noise. This is the stochastic heat equation with additive noise. This is a canonical example of a class of growth processes which share the following features:

1. Local smoothening
2. Random forcing by noise which decorrelates in space-time.

This are known as the **Edwards-Wilkinson universality class**.

Example 1.2. The situation changes completely if we add bias. In the SSEP, the random walks were symmetric. Consider a situation where a particle jumps to the right with probability p and jumps to the left with probability q, where $p \neq q$. This is known as an **asymmetric exclusion process (ASEP)**. In the **totally asymmetric exclusion process (TASEP)**, $p = 1$ ad $q = 0$. This is equivalent to what is known as the **corner growth model**. The TASEP evolves as follows:



What is the PDE governing the evolution? This is the Burgers' equation: If we write $\rho := \frac{dh}{dx}$, the equation is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\rho(1-\rho).$$

The key point is that this is a nonlinear PDE. A similar nonlinear PDE will appear as long as $p \neq q$. These belong to the **Kardar-Parisi-Zhang universality class**, which contains processes that exhibit

1. Local smoothing
2. Random focing
3. Local growth depending nonlinearly on the gradient.

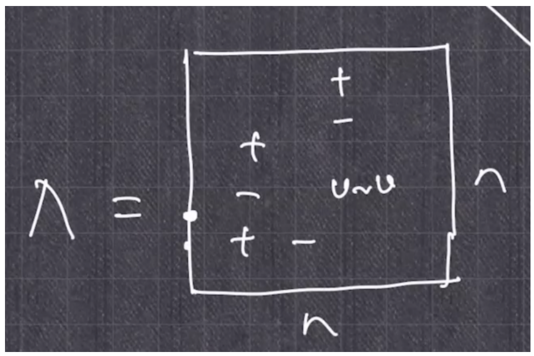
It turns out that the evolution can be described by the PDE

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial_x^2} h + \underbrace{(\nabla h)^2}_{\text{nonlinear flucuation}} + \eta,$$

We will encounter non-Gaussian fluctuations such as Airy$_2$ processes, Ferrari Spohn diffusions, and more.

1.3 The Ising model

Example 1.3. In the theory of spin systems, the Ising model is a model of ferro-magnetism. We have a lattice of sites, each of which can take the value $+1$ or -1. So our state space is $\{\pm 1\}^\Lambda$, where Λ is an $n \times n$ lattice.

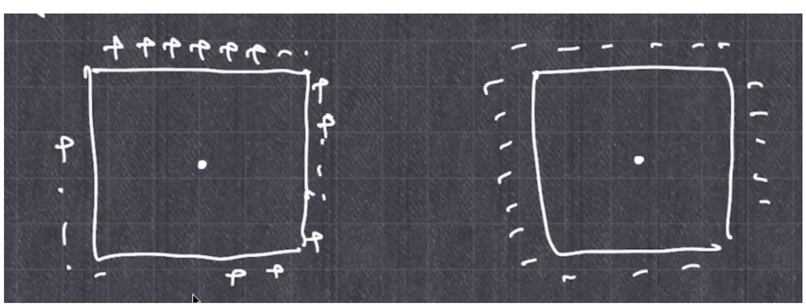


Then we consider the measure on this state space given by

$$\mathbb{P}(\sigma) \propto \exp\left(-\beta \sum_{u \sim v} \mathbb{1}_{\{\sigma_u \neq \sigma_v\}}\right), \qquad \sigma \in \{\pm 1\}^\Lambda,$$

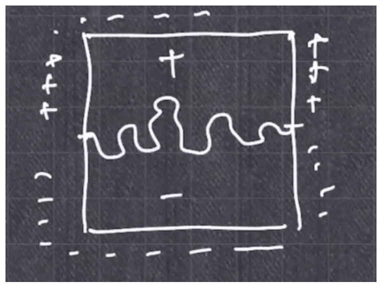
where β is known as the **inverse temperature**. We will also consider a magnetic field λ eventually.

It is well known that the Ising model exhibits a phase transition in β. If we look at two different boundary conditions, we get two different measures for what the spin is at the center σ_0 of the lattice:



What happens when $\beta = 0$? Then $\sigma_0 = \pm 1$ with probability 1/2. What about happens when $\beta = \infty$? With the + boundary condition, we have $\sigma_0 = 1$ with probability 1, and with the − boundary condition, we have $\sigma_0 = -1$ with probability 1. There is a critical β_c such that for $\beta > \beta_c$, the boundary has an effect uniformly in the box size and for $\beta < \beta_c$, the effect of the boundary decays exponentially in the box size.

If we take $\beta > \beta_c$, look at a boundary condition which is + on the top half and − on the bottom half. We will have the + phase mostly on top and the - phase on the bottom. There will be an interface between these two phases.



Here is a simulation: